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SOME PROBLEMS OF TREND IDENTIFICATION AND ESTIMATION

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Abstract. In the paper are formulated some problems that can occur by estimation of the time series trends. Further are proposed some trend's properties and formulated some theoretical questions which answers can simplify trend identification.

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1 Introduction

Classic time series analyse works with model [1]:

$$X_t = T_t + S_t + C_t + R_t \qquad (1.1)$$

where
$T_t =$ trend value at time t
$S_t =$ seasonal value at time t
$C_t =$ cyclical value at time t
$R_t =$ irregular, or random, value at time t

Some other types (e.g. multiplicative time series with positive observations) can be transformed into this model. Trend component (T_t) describes long term behaviour. Other components (S_t and C_t) describe short term behaviour. For behaviour, that cannot be described by components above is used irregular, or random, component (R_t). When properties of components are known, it is easier to estimate them. If properties are unknown it is necessary to use some other techniques to detach them. In our paper, we will discuss the trend detection and we will consider simplified model:

$$X_t = T_t + R_t \tag{1.2}$$

For the trend estimation is often used linear model:

$$T_t = vt + s \tag{1.3}$$

where
$v =$ trend slope
$s =$ trend intercept

The problem of the model (1.3) is inconsistent time composition. This fact is demonstrated in Fig. 1. There are linear trends of a currency EUR in CZK – data of the year 2005, data of the year 2006 and data of both years. It is obvious, that composition of the first trend (2005) and the second one (2006) is not the trend of both years.

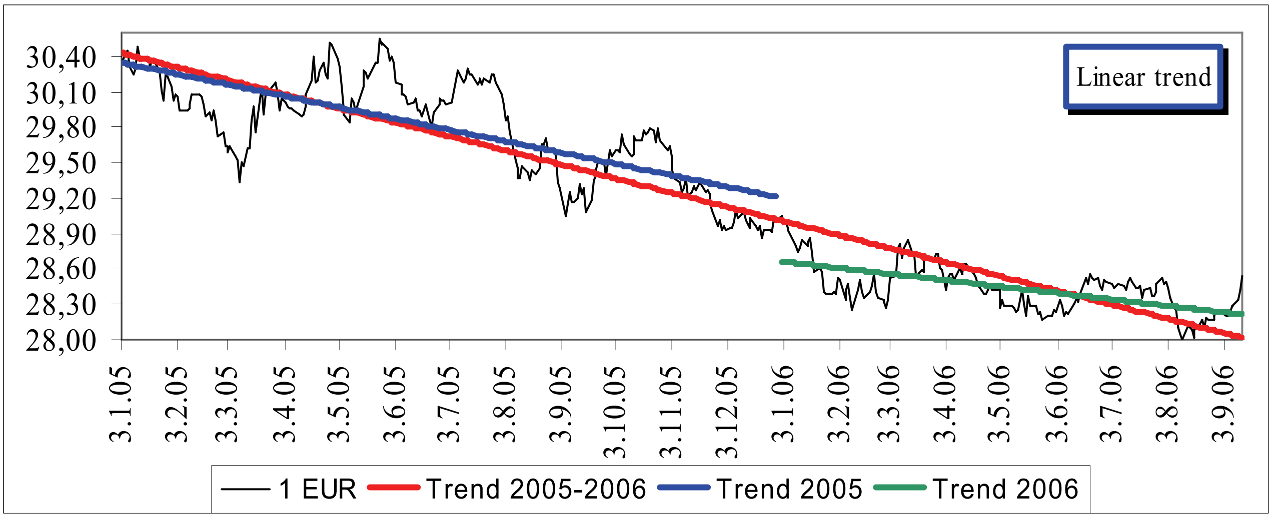

Fig. 1: Time composition

2 Consistence of the Trend Estimation

First, we will define what is meant by time composition. Let $x_{<t_1,t_2>}(t)$ be a function on an interval $<t_1,t_2>$ and let $y_{<t_2,t_3>}(t)$ be function on an interval $<t_2,t_3>$, where $t_1 < t_2 < t_3$. Time composition of $x_{<t_1,t_2>}(t)$ and $y_{<t_2,t_3>}(t)$ will be function $z_{<t_1,t_3>}(t)$:

$$z_{<t_1,t_3>}(t) = x_{<t_1,t_2>}(t) \circ y_{<t_2,t_3>}(t) = \begin{cases} x_{<t_1,t_2>}(t) & \Leftrightarrow t \in <t_1,t_2) \\ 0{,}5 \cdot \left(x_{<t_1,t_2>}(t) + y_{<t_2,t_3>}(t) \right) & \Leftrightarrow t = t_2 \\ y_{<t_2,t_3>}(t) & \Leftrightarrow t \in (t_2,t_3> \end{cases} \tag{2.1}$$

The segmentation of the function $z_{<t_1,t_3>}(t)$, for each $t_d \in <t_1,t_3>$ is pair of functions:

$$x_{<t_1,t_d>}(t) = z_{<t_1,t_3>}(t) \Leftrightarrow t \in <t_1,t_d>$$
$$y_{<t_d,t_3>}(t) = z_{<t_1,t_3>}(t) \Leftrightarrow t \in <t_d,t_3> \tag{2.2}$$

Definition 2.1: Algorithms for estimation course (trend, function) $x_{<t_1,t_2>}(t)$ on subintervals of $<t_1,t_2>$ are consistent, if $\forall t_d \in <t_1,t_2>$: $x_{<t_1,t_2>}(t) = x_{<t_1,t_d>}(t) \circ x_{<t_d,t_2>}(t)$.

Remark 2.1: It is obvious that algorithm for linear trend estimation on finite interval is not consistent.

Remark 2.2: Algorithms using a moving average:

$$T_{\pm k}(t) = \frac{1}{2k+1}\sum_{i=-k}^{k} x(t+i) \tag{2.3}$$

for a trend estimation are consistent on the integral interval of time indexes $<t_0,t_n>$, when we have observations of $x(t)$ from interval $<t_0-k,t_n+k>$. This is demonstrated in Fig. 2, where is the same example as in Fig. 1 – a currency EUR in CZK. Trend is estimated by moving average with $k=30$. When we estimate the trend for any subinterval, we get always the same course. Problem of this estimate is that we do not get the course in the first and in the last 30 days.

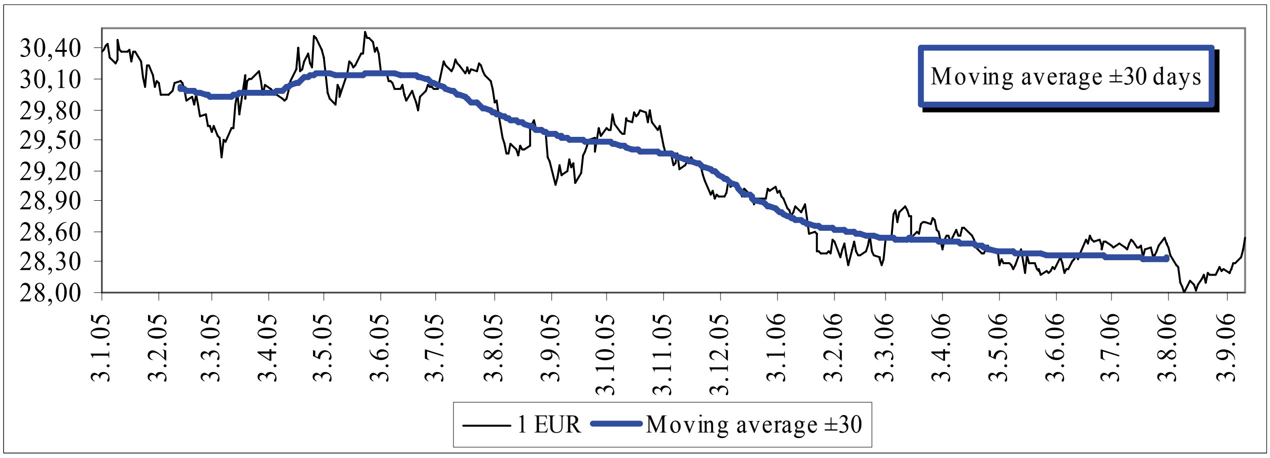

Fig. 2 Moving average

Remark 3: Algorithms for calculation of the linear (even nonlinear) regression almost never lead on the consistent trends. On the other side, algorithms using some filters lead on the consistent trends.

Remark 4: Consistence of the trend depends on data – when a linear trend is computed for data which are placed on the straight line, the trend estimate will be also consistent. The term of consistence could be generalized to solve this, but we don't need it in this paper. Terminologically: The trend is understood as an algorithm over data.

3 Some Linear Filters as Algorithms for Trend Computation

It is obvious that when we compute with discrete time axis $Z = (-\infty, ..., -2, -1, 0, 1, 2, ..., +\infty)$, then every algorithm for generating the trend in the form:

$$T(t) = \sum_{i=-\infty}^{+\infty} w(i)x(t-i) = \sum_{i=-\infty}^{+\infty} w(t-i)x(i), \quad i \in Z \tag{3.1}$$

where
$x(i)$ = observations
$w(i)$ = weights

leads to consistent course. Not all filters (sequence of weights $w(i);\ i \in Z$) correspond with the requirements of the trend. Therefore, filters have to fulfil following requirements:

Requirement 3.1: For bounded sequence $x(i);\ i \in Z$ should the filter generate bounded sequence $T(t)$. Necessary and sufficient condition [3] for stability of the filter (3.1) is:

$$\sum_{i=-\infty}^{+\infty} |w(i)| = W < +\infty \tag{3.2}$$

Requirement 3.2: The filter should preserve the constant data, i.e.:

$$K = \sum_{i=-\infty}^{+\infty} w(t-i)K;\ \forall K \in R_1 \tag{3.3}$$

Necessary and sufficient condition for (3.3) is:

$$1 = \sum_{i=-\infty}^{+\infty} w(i) \tag{3.4}$$

Requirement 3.3: The filter should have a "finite memory" (forward and backward). A sufficient condition for this requirement is a finite count of nonzero terms of the sequence $w(i);\ i \in Z$. This condition is not necessary – e.g. exponential filter

$$T(i) = (1-\alpha)T(i-1) + \alpha\, x(i) \tag{3.5}$$

is filter with finite memory and infinite weight sequence.

Those are requirements for deterministic behaviour of the filter. Now we will study stochastic properties – in the next part $x(i);\ i \in Z$ will be the stochastic sequence over the integral support (Z). We will work with FIR filters (final impulse response, i.e. filters that fulfil requirement 3.3) and by this way we will simplify problems with existence and stability of filters.

The mean value of the trend generated by filter is:

$$E\{T(t)\} = \sum_{i=-\infty}^{+\infty} w(t-i)E\{x(i)\} \tag{3.6}$$

The deviation between a realization and a mean value is:

$$T(t) - E\{T(t)\} = \sum_{i=-\infty}^{+\infty} w(t-i)\left(x(i) - E\{x(i)\}\right) \tag{3.7}$$

For stable filters (with absolutely convergent weight sequence) is possible to define the autocovariance function, let

$$\left(T(s) - E\{T(s)\}\right)\left(T(t) - E\{T(t)\}\right) = \sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} w(t-i)w(s-j)\left(x(i) - E\{x(i)\}\right)\left(x(j) - E\{x(j)\}\right) \tag{3.8}$$

When we apply the operator of the mean value on (3.8), we get:

$$Cov_T(s,t) = \sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} w(t-i)w(s-j)Cov_x(i,j) \tag{3.9}$$

If $s = t$ we get a skedasticity function:

$$\sigma_T^2(t) = \sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} w(t-i)w(t-j)Cov_x(i,j) \tag{3.10}$$

When is on the input of the filter white noise (i.e. first and second moment are stationary, the sequence is uncorrelated), the skedasticity function is $\sigma_{Tw}^2(t) = \sum_{i=-\infty}^{+\infty} w(t-i)^2 \sigma_w^2 = \sigma_w^2 \sum_{i=-\infty}^{+\infty} w(t-i)^2$.
The goal is to have filter generating a trend with a long-time, slowly course, so we want to be $\sum_{i=-\infty}^{+\infty} w(t-i)^2 < 1$. From this expression we get next requirement.

Requirement 3.4: The impulse response of the filter should fulfil:

$$\sum_{i=-\infty}^{+\infty} w(i)^2 < 1 \tag{3.11}$$

Remark 3.5: When the filter fulfils the requirement 3.4, it also fulfils the requirement 3.1 (i.e. stability of the filter).

By the same way as in autocovariance function, we get short-time trend variability:

$$E\{(T(t+1)-T(t))^2\} = \sum_{i=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty} w(t+1-i)w(t-j)E\{x(i)x(j)\} \tag{3.12}$$

When a white noise with zero mean value is on the input, we get $E\{(T(t+1)-T(t))^2\} = \sigma_w^2 \sum_{i=-\infty}^{+\infty} w(t+1-i)w(t-i)$, for such sequence also holds $E\{(x(t+1)-x(t))^2\} = 2\sigma_w^2$ and the ratio $\frac{E\{(T(t+1)-T(t))^2\}}{E\{(x(t+1)-x(t))^2\}}$ is in this occasion:

$$sm(t) \overset{def}{=} \frac{E\{(T(t+1)-T(t))^2\}}{E\{(x(t+1)-x(t))^2\}} = \frac{\sigma_w^2 \sum_{i=-\infty}^{+\infty} w(t+1-i)w(t-i)}{2\sigma_w^2} = 0.5 \sum_{i=-\infty}^{+\infty} w(t+1-i)w(t-i) \tag{3.13}$$

This expression we will call efficiency of the short-time variability muting. For the filters type of moving average – i.e.:

$$\begin{aligned} w(i) &= \frac{1}{L+K+1} \quad \Leftrightarrow \; i \in < -L, K >;\; L, K \geq 0;\; -L \leq K; \\ w(i) &= 0 \quad \Leftrightarrow i \notin < -L, K > \end{aligned} \tag{3.14}$$

has the expression (3.13) this form:

$$sm(t) = \frac{(L+K)}{(L+K+1)^2} \tag{3.15}$$

The expression (3.13) (or in case of the moving average expression (3.15)) we could understand as a comparative standard in the meaning of the worst possible case. The expression

$$sm(t) \overset{def}{=} \frac{E\{(T(t+1)-T(t))^2\}}{E\{(x(t+1)-x(t))^2\}} \tag{3.16}$$

has statistical analogy:

$$\overline{sm(T)} = \frac{\sum_{t=0}^{M} \{(T(t+1)-T(t))^2\}}{\sum_{t=0}^{M} \{(x(t+1)-x(t))^2\}} \tag{3.17}$$

which represents a measure of the non-elimination of the residual, “random”, irregular component. In this expression M means maximal time for which we have course and trend.

4 Some Examples of the Trends as a Production of Filtration

Example 4.1: Moving average

$$T_d(t) = \frac{1}{2d+1}\sum_{i=-d}^{+d} x(t+i) \tag{4.1}$$

Example 4.2: Max-Min filter

$$T(t) = 0.5\left(\max_{i=-d,\dots,+d} x(t+i) + \min_{i=-d,\dots,+d} x(t+i)\right) \tag{4.2}$$

Example 4.3: Filter of the moving regression

$$T(t) = a_{<t-d,t+d>} t + b_{<t-d,t+d>} \tag{4.3}$$

where

$a_{<t-d,t+d>}$ = coefficient of the scale computed by algorithm of linear regression from data $\langle x(t-d); x(t+d)\rangle$

$b_{<t-d,t+d>}$ = parameter of the location computed by algorithm of linear regression from data $\langle x(t-d); x(t+d)\rangle$

Comparison of all three non-linear filters is in Fig. 3. It is still the same example as in two previous cases – a currency EUR in CZK. Trends are gained from ±30 days. The measure of the non-elimination of the irregular component (3.17) is:

$$sm(\text{moving average}) = 0{,}82\%$$
$$sm(\text{max-min filter}) = 4{,}17\%$$
$$sm(\text{moving regression}) = 1{,}20\%$$

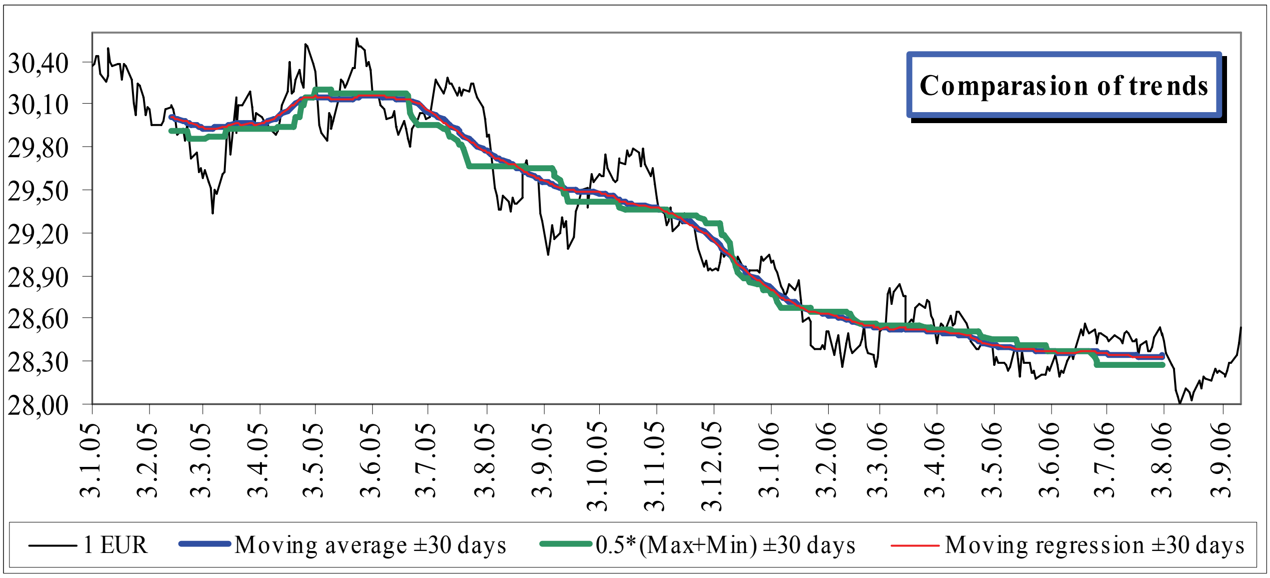

Fig. 3: Demonstration of three filters

From the demonstration is obvious, that the measure of non-elimination of the irregular component (3.17) is the right measure. When we want to set its suitable values, we have to use some calibrations procedure (in many cases on the experimental base). In the case of the linear filters, we could use frequency procedure (low pass filters) too, we refer to [3].

5 Resume and Suggestions

In this work was defined the term of consistent algorithm for trend detection. It was said, that direct filters (type of MA, linear or even nonlinear analogue) which fulfil defined restrictions represent consistent algorithms. The measure of the non-elimination of the irregular component was proposed by expression (3.17) – it seams like a suitable measure of "extraction" of the trend from time series. This work is subject for further investigation and searching solution of following problems:

- Which conditions must be satisfied for AR filters $T(t)=\sum_{\substack{i=-d_1\\ i\neq 0}}^{d_2} a_i T(t+i)+bx(t);\ d_1, d_2 \geq 0$ to be consistent algorithms for the trend determination. Solution isn't perhaps difficult in the field of sufficient conditions and it will be connected with invertibility of the filter (conversion from type MA on AR and backward)? Some ARMA type filters $T(t)=\sum_{\substack{i=-d_1\\ i\neq 0}}^{d_2} a_i T(t+i)+\sum_{i=-d3}^{+d4} x(t+i);\ d_1, d_2, d_3, d_4 \geq 0$ could be consistent algorithms (e.g. it is possible to transform moving average to them).

- Is it possible to set up a priory property of the trend (when it isn't deterministic course)? General answer is "yes" – e.g. with help of the filter of low-pass filter. Specified problem: Is it possible without using frequency terms – e.g. to set up minimal and maximal value of derivation, with condition of existence and continuity of first derivation?

References

[1] CIPRA, T.: Analýza časových řad s aplikacemi v ekonomii, SNTL, Alfa, Praha, Bratislava 1986. In Czech.

[2] ARLT J.: Moderní metody modelování ekonomických časových řad, GRADA Publishing, Praha 1999. In Czech.

[3] PROUZA L.: Úvod do teorie a aplikací lineárních impulsních soustav. ACADEMIA, Praha 1967. In Czech.

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