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DISTORTION OF PROBABILITY MODELS

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Abstract. The proposed paper deals with one of possible methods for modelling the relation of two probability models at distribution functions level. The essence of the used concept is a distortion of the distribution function. Distortion function is a non-decreasing mapping of the interval [0,1] into itself. Some transformation effects of the distribution function of distortion functions are derived in this contribution. The exploitation of the distortion function concept of probability model quality measurement is also proposed. Practical usage is represented by the transformation of empirical distribution function of weekly forecast error of PX50 index value to a normal (Gaussian) model.

Key words. Distribution, distortion, density, divergence

1 Motivation

Often used means of probability modelling and statistical evaluating is a transformation of a random variable. The exploitation of transformation procedures of distribution functions and densities is less frequent. Such procedures are used in risk measurement, insurance theory and also in other fields [1,2,3,4]. Our contribution attempts to suggest the link between this concept and classical information theory [5,6].

2 Conception

Let's denote:

$F(x)$ is a distribution function,
$f(x)$ is a density of a random variable,
$F_D(x)$ is another distribution function,
$f_D(x)$ is its corresponding density.

Let us have a function $G(y)$ called a distortion function on condition that:
M1: $G(y)$ is a non-decreasing function on the interval [0,1],

M2: $G(0) = 0$, $G(1) = 1$,

$$g(y) = \frac{d}{dy} G(y)$$

M3: except a finite number of points, exists on the interval [0,1].

And for which it holds:

$$F_D(x) = G(F(x)) \tag{1}$$

Remark: The function *G(y)* is a distribution function on the interval [0,1] and so *g(y)* is its density. Therefore

$$f_D(x) = g(F(x)) f(x) \tag{2}$$

except a finite number of points.
It is obvious that for every distribution function *F(x)* and for every function *G(y)* satisfying M1, M2 and M3, *G(F(x))* is again a distribution function.

The couple *F, f* can be interpreted as a distribution function and its density, and F_D, f_D as some of their estimates. Then it is possible to "measure the distance", (e.g.) by the divergence [5] (for simplicity we will suppose that *F(x)* is increasing and continuous):

$$D(f \,\|\, f_D) = \int_{-\infty}^{+\infty} f(x) \lg \frac{f(x)}{f_D(x)} dx = \int_{-\infty}^{+\infty} f(x) \lg \frac{f(x)}{g(F(x)) f(x)} dx = -\int_{-\infty}^{+\infty} f(x) \lg g(F(x)) dx$$

After substitution $y = F(x) \quad \Rightarrow \quad dy = f(x)dx \quad \Leftrightarrow \quad dx = \frac{1}{f(x)} dy$ we get:

$$D(f \,\|\, f_D) = -\int_0^1 \lg g(y) dy . \tag{3}$$

Likewise:

$$D(f_D \,\|\, f) = \int_{-\infty}^{+\infty} f_D(x) \lg \frac{f_D(x)}{f(x)} dx = \int_{-\infty}^{+\infty} g(F(x)) f(x) \lg \frac{g(F(x)) f(x)}{f(x)} dx =$$

$$= \int_{-\infty}^{+\infty} g(F(x)) f(x) \lg g(F(x)) dx .$$

After substitution $y = F(x) \quad \Rightarrow \quad dy = f(x)dx \quad \Leftrightarrow \quad dx = \frac{1}{f(x)} dy$ we get:

$$D(f_D \,\|\, f) = \int_0^1 g(y) \lg g(y) dy = -H(g) \tag{4}$$

$$J(f_D, f) = D(f_D \,\|\, f) + D(f \,\|\, f_D) = -\int_0^1 \lg g(y) dy - H(g) = \int_0^1 (g(y) - 1) \lg g(y) dy$$

(symmetrical divergence [5,6]). So the "difference" between the two probability descriptions depend only on the "density" of distortion.

3 Moments of the distortion

For continuous and increasing distribution function $F(x)$, we can easily determine moments of a random variable X that follows the distribution function $G(F(x))$. If $E_{G(F(*))}\{X^a\}$ exists, then:

$$E_{G(F(*))}\{X^a\} = \int_{-\infty}^{+\infty} x^a f_D(x)dx = \int_{-\infty}^{+\infty} x^a g(F(x))f(x)dx =$$

$$= \int_0^1 (F^{-1}(x))^a g(x)dx = E_{G(*)}\{(F^{-1}(X))^a\},$$

where $F^{-1}(x)$ is a generalized inversion of the distribution function $F(x)$ and $(F^{-1}(x))^a$ is its rational power.

In particular, for $a = 1, 2$:

$$E_{G(F(*))}\{X\} = \int_0^1 (F^{-1}(x))g(x)dx \text{ and } E_{G(F(*))}\{X^2\} = \int_0^1 (F^{-1}(x))^2 g(x)dx\text{, respectively,} \tag{5}$$

if they exist, and in such a case we get:

$$\sigma^2_{G(F(*))}\{X\} = \int_0^1 (F^{-1}(x))^2 g(x)dx - \left(\int_0^1 (F^{-1}(x))g(x)dx\right)^2. \tag{6}$$

4 Some types of distortion functions

A wide range of parametric families of distortion functions is mentioned in [3]. In this paper, we will deal with two of them:

a) proportional: $G(y) = y^{\alpha} \;\Rightarrow\; g(y) = \alpha\, y^{\alpha-1};\; \alpha > 0$,

b) exponential: $G(y) = \dfrac{1-e^{-\lambda y}}{1-e^{-\lambda}} \;\Rightarrow\; g(y) = \dfrac{\lambda e^{-\lambda y}}{1-e^{-\lambda}};\; \lambda \neq 0$.

For the latter one, it holds:

$$\lim_{\lambda\to 0} G(y) = \lim_{\lambda\to 0}\frac{1-e^{-\lambda y}}{1-e^{-\lambda}} = \lim_{\lambda\to 0}\frac{ye^{-\lambda y}}{e^{-\lambda}} = y,$$

which is an identical distortion function.

For the proportional distortion it holds that:

$$D(f_D \,||\, f) = \lg\alpha - \frac{\alpha-1}{\alpha} \text{ and } D(f \,||\, f_D) = -\lg\alpha + \alpha - 1 \text{ and}$$

$$J(f_D, f) = D(f_D \,||\, f) + D(f \,||\, f_D) = \frac{(\alpha-1)^2}{\alpha}.$$

For the exponential distortion it holds that:

$$D(f_D \,||\, f) = \lg\frac{\lambda}{1-e^{-\lambda}} + (1+\lambda)\frac{e^{-\lambda}}{1-e^{-\lambda}} - \frac{1}{1-e^{-\lambda}} \text{ and } D(f \,||\, f_D) = -\lg\frac{\lambda}{1-e^{-\lambda}} + \frac{\lambda}{2} \text{ and}$$

$$J(f_D, f) = D(f_D \,||\, f) + D(f \,||\, f_D) = \frac{\lambda}{2} + (1+\lambda)\frac{e^{-\lambda}}{1-e^{-\lambda}} - \frac{1}{1-e^{-\lambda}}.$$

5 Some applications and usage

Figs. 1, 2, 3 present the family of exponential distortions.

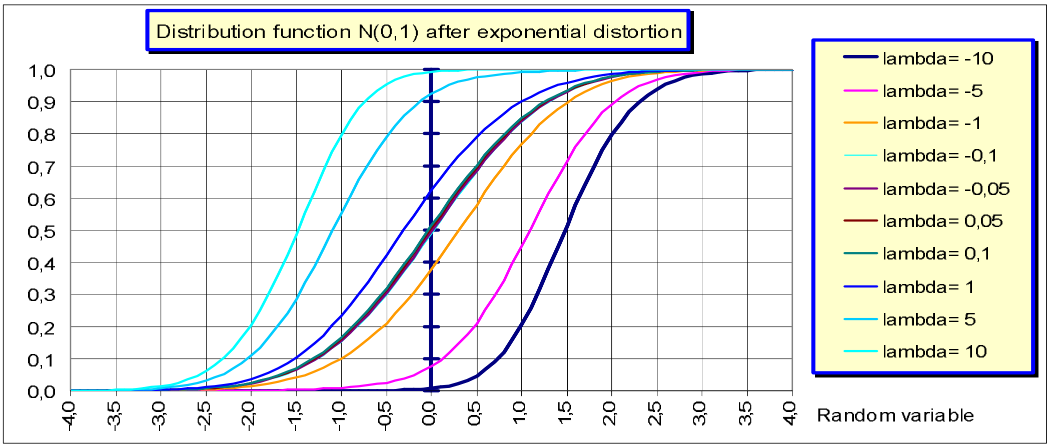

Fig. 1. Distribution N(0,1) after exponential distortion.

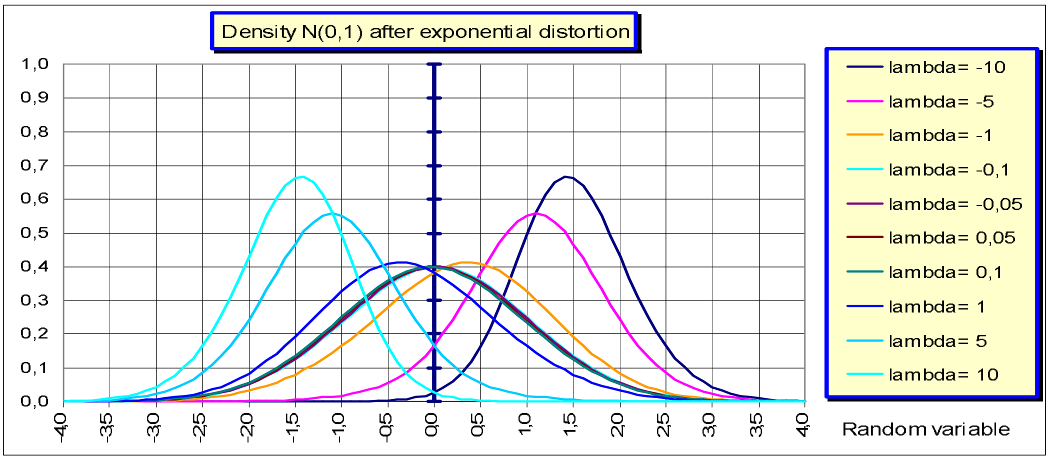

Fig. 2. Density N(0,1) after exponential distortion.

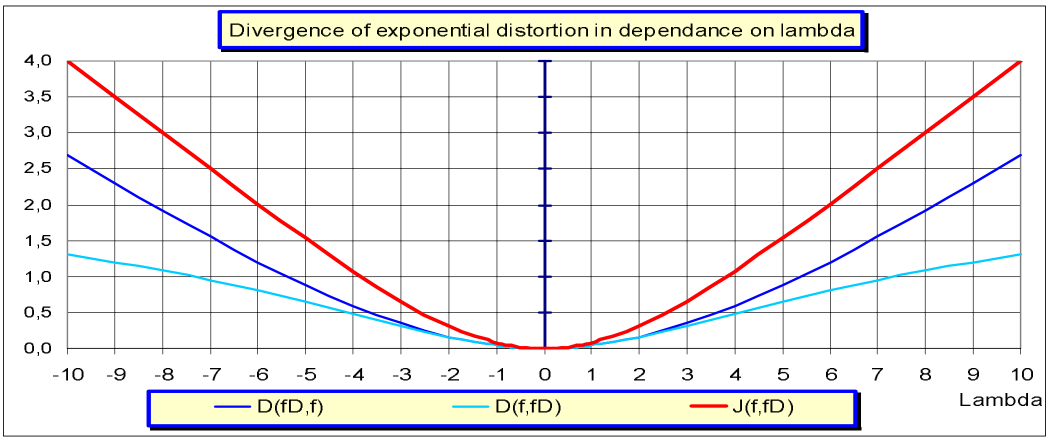

Fig. 3. Divergence measures N(0,1) after exponential distortion.

Possible exploitation of the proposed concept is modelling the distortion of empirical distribution function of PX50 index estimation error distribution up to one week with the use of predictor with „one-year memory“ to normal probability distribution, and conversely, see Figs. 4, 5, 6.

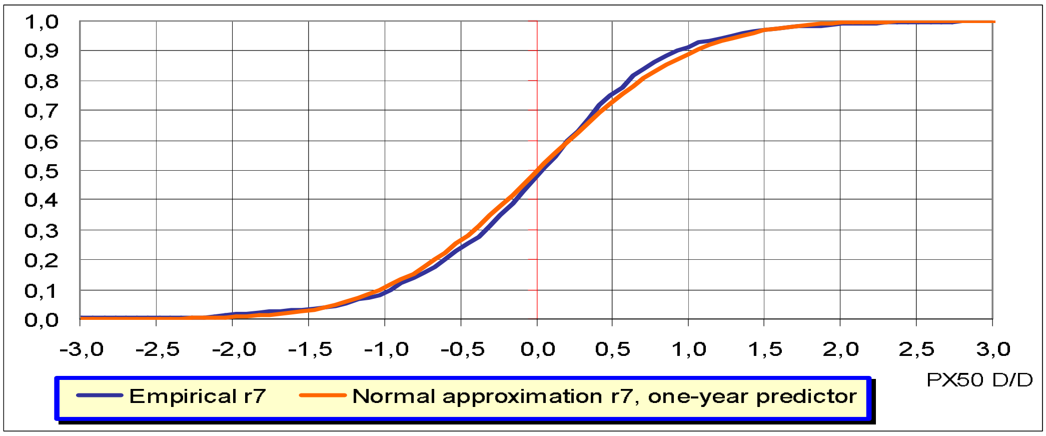

Fig. 4. Empirical distribution and its normal approximation (r7, one-year predictor).

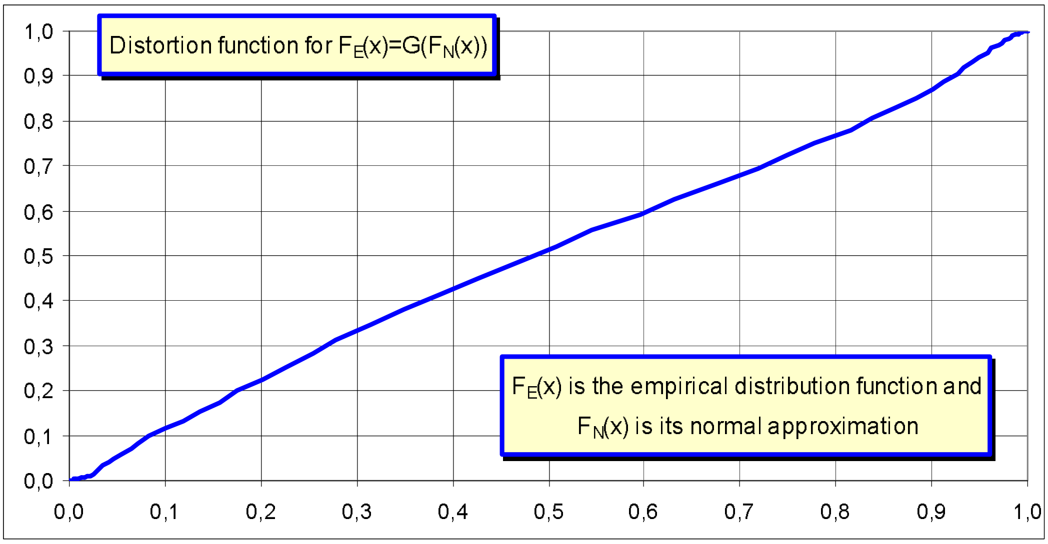

Fig. 5. Distortion from normal to empirical (r7, one-year predictor).

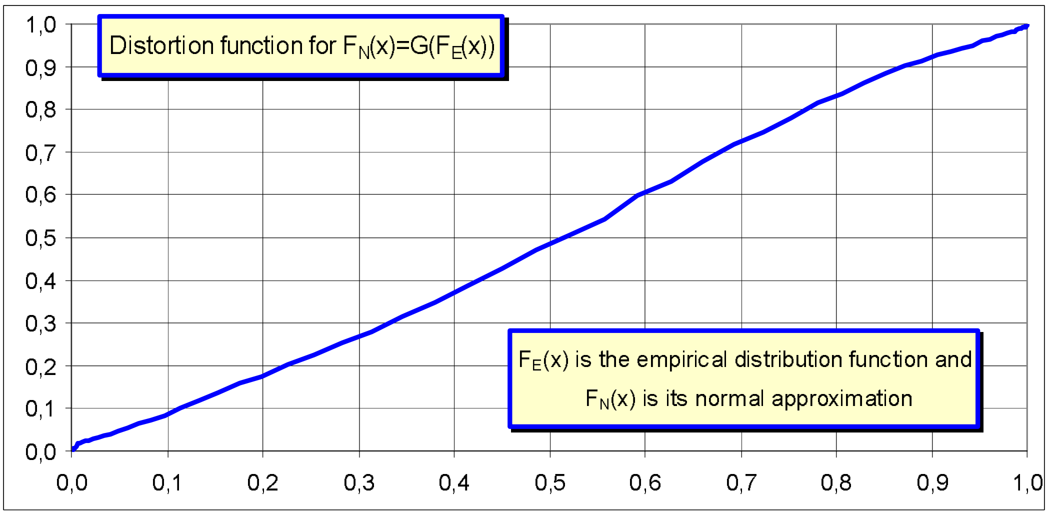

Fig. 6. Distortion from empirical to normal (r7, one-year predictor).

6 Conclusions and further progress

The mentioned relations were derived with strict assumptions. It is obvious that such assumptions were used for simplicity of proofs. It is also evident that the presented method can be used with more free assumptions which are also more verifiable in real life. It could be also interesting to investigate possible duality between transformation (distortion) of distribution function and transformation of variables. Some transformations are possible as well. See the example in Fig. 7, where a distortion of normed normal probability distribution into a normed exponential distribution appears.

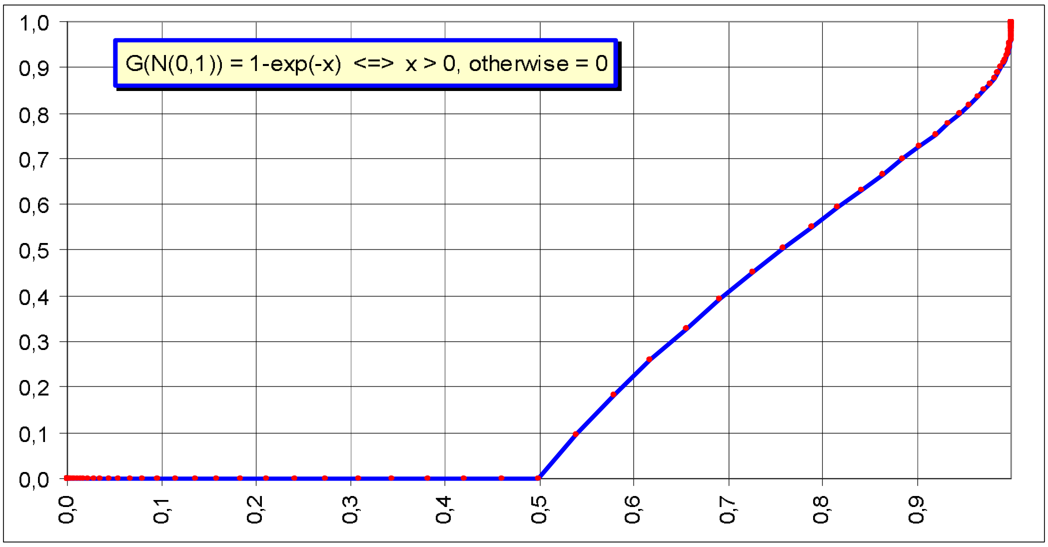

Fig. 7. Distortion from "normed" normal probability distribution to "normed" exponential one.

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