

LECTURE 23: Hidden Markov Models

- Discrete Markov Processes
- Hidden Markov Models
- Illustrative examples
- Forward and Backward procedures
- The Viterbi algorithm



Introduction

- **The next two lectures in the course deal with the recognition of temporal or sequential patterns**
 - Sequential pattern recognition is a relevant problem in a number of disciplines
 - Human-computer interaction: Speech recognition
 - Bioengineering: ECG and EEG analysis
 - Robotics: mobile robot navigation
 - Bioinformatics: DNA base sequence alignment
- **A number of approaches can be used to perform time series analysis**
 - **Tap delay lines** can be used to form a feature vector that captures the behavior of the signal during a fixed time window
 - This represents a form of “short-term” memory
 - This simple approach is, however, limited by the finite length of the delay line
 - **Feedback connections** can be used to produce recurrent MLP models
 - Global feedback allows the model to have “long-term” memory capabilities
 - Training and using recurrent networks is, however, rather involved and outside the scope of this class (refer to [Principe et al., 2000; Haykin, 1999])
 - Instead, we will focus on **Hidden Markov Models**, a statistical approach that has become the “gold standard” for time series analysis



Discrete Markov Processes (1)

■ Consider a system described by the following process

- At any given time, the system can be in one of N possible states $S=\{S_1, S_2, \dots, S_N\}$
- At regularly spaced times, the system undergoes a transition to a new state
- Transition between states can be described probabilistically

■ In general, the probability that the system is in state $q_t=S_j$ will be a function of the complete history of the system

- To simplify the analysis, however, it is common to assume that the state of the system at time t depends only on the state at time $t-1$

$$P(q_t = S_j | q_{t-1} = S_i, q_{t-2} = S_k, \dots) = P(q_t = S_j | q_{t-1} = S_i)$$

- This is known as a **first-order Markov Process**

- Higher-order Markov Processes assume a dependence at larger lags ($t-2, t-3, \dots$)
- We will also assume that the transition probability between any two states is independent of time

$$a_{ij} = P(q_t = S_j | q_{t-1} = S_i)$$

- Subject to
$$\begin{cases} a_{ij} \geq 0 \\ \sum_{j=1}^N a_{ij} = 1 \end{cases}$$

These lecture notes were produced following [Rabiner and Juang, 1993]



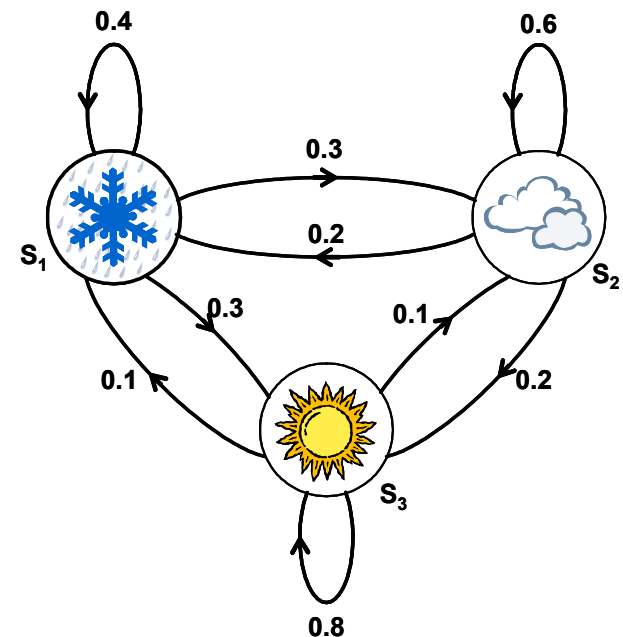
Discrete Markov Processes (2)

■ To illustrate these concepts consider a simple three-state Markov model of the weather

- Any given day, the weather can be described as being
 - State 1: Precipitation (rain or snow)
 - State 2: Cloudy
 - State 3: Sunny
- Transitions between states are described by the transition matrix

$$A = \{a_{ij}\} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

- This model can then be described by the following directed graph



Discrete Markov Processes (3)

■ Question

- Given that the weather on day $t=1$ is sunny, what is the probability that the weather for the next 7 days will be “sun, sun, rain, rain, sun, clouds, sun” ?

■ Answer:

$$\begin{aligned} &P(S_3, S_3, S_3, S_1, S_1, S_3, S_2, S_3 | \text{model}) = \\ &P(S_3)P(S_3 | S_3)P(S_3 | S_3)P(S_1 | S_3)P(S_1 | S_1)P(S_3 | S_1)P(S_2 | S_3)P(S_3 | S_2) = \\ &\pi_3 a_{33} a_{33} a_{13} a_{11} a_{31} a_{23} a_{32} = 1(0.8)(0.8)(0.1)(0.4)(0.3)(0.1)(0.2) \end{aligned}$$

■ Question

- What is the probability that the weather stays in the same known state S_i for exactly T consecutive days?

■ Answer:

$$P(q_t = S_i, q_{t+1} = S_i, q_{t+2} = S_i, \dots, q_{t+T-1} = S_i, q_{t+T} = S_{j \neq i}) = a_{ii}^{T-1}(1 - a_{ii})$$



Discrete Markov Processes (4)

- **The previous model assumes that each state can be uniquely associated with an observable event**
 - Once an observation is made, the state of the system is then trivially retrieved
 - This model, however, is too restrictive to be of practical use for most realistic problems
- **To make the model more flexible, we will assume that the outcomes or observations of the model are a probabilistic function of each state**
 - Each state can produce a number of outputs according to a unique probability distribution, and each distinct output can potentially be generated at any state
 - These are known as **Hidden Markov Models (HMM)**, because the state sequence is not directly observable, it can only be approximated from the sequence of observations produced by the system



The coin-toss problem (1)

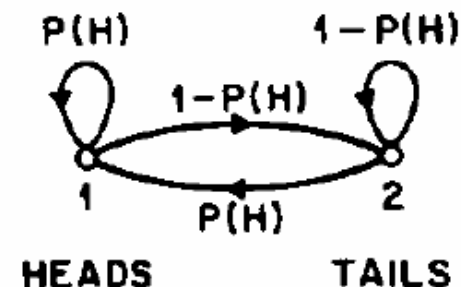
- **To illustrate the concept of an HMM consider the following scenario**
 - Assume that you are placed in a room with a curtain
 - Behind the curtain there is a person performing a coin-toss experiment
 - This person selects one of several coins, and tosses it: heads (H) or tails (T)
 - The person tells you the outcome (H,T), but not which coin was used each time
- **Your goal is to build a probabilistic model that best explains a sequence of observations $O=\{o_1, o_2, o_3, o_4, \dots\}=\{H, T, T, H, \dots\}$**
 - The coins represent the states; these are hidden because you do not know which coin was tossed each time
 - The outcome of each toss represents an observation
 - A “likely” sequence of coins may be inferred from the observations, but this state sequence will not be unique
- **If the coins are hidden, how many states should the HMM have?**



The coin-toss problem (2)

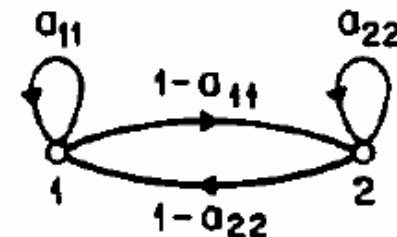
■ One-coin model

- In this case, we assume that the person behind the curtain only has one coin
- As a result, the Markov model is observable since there is only one state
- In fact, we may describe the system with a deterministic model where the states are the actual observations (see figure)
- In either case, the model parameter $P(H)$ may be found from the ratio of heads and tails



■ Two-coin model

- A more sophisticated HMM would be to assume that there are two coins
 - Each coin (state) has its own distribution of heads and tails, to model the fact that the coins may be biased
 - Transitions between the two states model the random process used by the person behind the curtain to select one of the coins
- The model has four free parameters



$$\begin{array}{ll} P(H) = P_1 & P(H) = P_2 \\ P(T) = 1 - P_1 & P(T) = 1 - P_2 \end{array}$$

From [Rabiner, 1989]



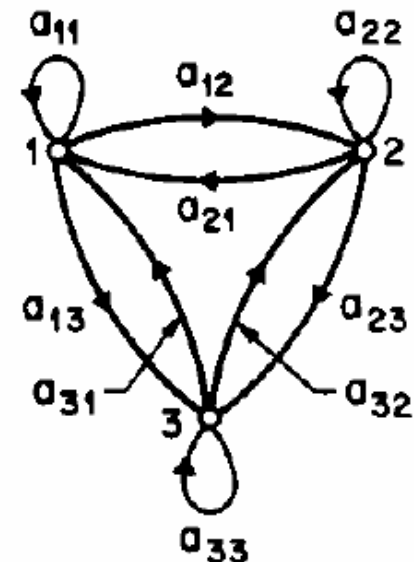
The coin-toss problem (3)

■ Three-coin model

- In this case, the model would have three separate states
 - This HMM can be interpreted in a similar fashion as the two-coin model
- The model has nine free parameters

■ Which of these models is best?

- Since the states are not observable, the best we can do is select the model that best explains the data (e.g., Maximum Likelihood criterion)
- Whether the observation sequence is long and rich enough to warrant a more complex model is a different story, though



	STATE		
	1	2	3
P(H)	P_1	P_2	P_3
P(T)	$1 - P_1$	$1 - P_2$	$1 - P_3$

From [Rabiner, 1989]



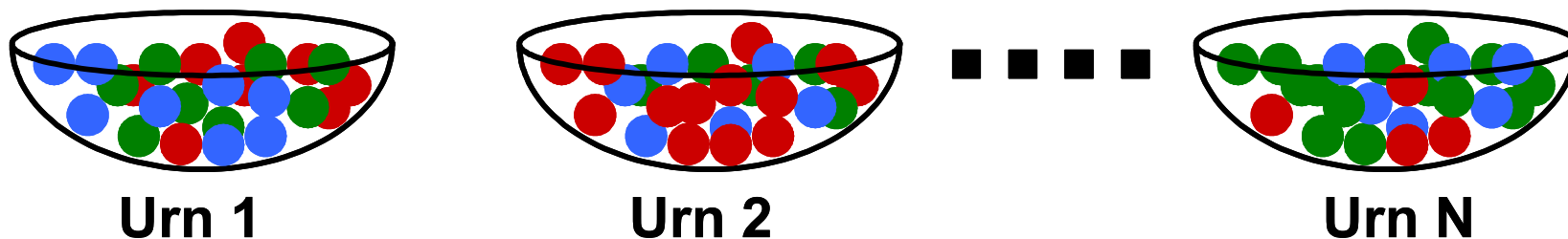
The urn-ball problem

■ To further illustrate the concept of an HMM, consider this scenario

- You are placed in the same room with a curtain
- Behind the curtain there are N urns, each containing a large number of balls with M different colors
- The person behind the curtain selects an urn according to an internal random process, then randomly grabs a ball from the selected urn
- He shows you the ball, and places it back in the urn
- This process is repeated over and over

■ Questions?

- How would you represent this experiment with an HMM?
- What are the states?
- Why are the states hidden?
- What are the observations?



Elements of a Hidden Markov Model

■ A Hidden Markov Model is characterized by the following

- N , the number of states in the model $S=\{S_1, S_2, \dots, S_N\}$
- M , the number of discrete observation symbols $V=\{v_1, v_2, \dots, v_N\}$
- $A=\{a_{ij}\}$, the state transition probability

$$a_{ij} = P(q_{t+1} = S_j | q_t = S_i)$$

- $B=\{b_j(k)\}$, the observation probability distribution

$$b_j(k) = P(o_t = v_k | q_t = S_j)$$

- π , the initial state distribution

$$\pi_j = P(q_1 = S_j)$$

■ Therefore, an HMM is specified by two scalars (N and M) and three probability distributions (A , B and π)

- In what follows, we will represent an HMM by the compact notation

$$\lambda = (A, B, \pi)$$



HMM generation of observation sequences

- **Given a completely specified HMM $\lambda=\{A,B,\pi\}$, how can an observation sequence $O=\{o_1,o_2,o_3,o_4,\dots\}$ be generated?**

1. Choose an initial state S_1 according to the initial state distribution π
2. Set $t=1$
3. Generate an observation o_t according to the observation distribution $b_j(k)$
4. Move to a new state S_{t+1} according to the state-transition distribution at that state a_{ij}
5. Set $t=t+1$ and return to 3 until $t \geq T$

- **Example**

- Generate an observation sequence with $T=5$ for a coin tossing experiment with three coins and the following probabilities

	S_1	S_2	S_3
P(H)	0.5	0.75	0.25
P(T)	0.5	0.25	0.75

$$A = \{a_{ij}\} = \frac{1}{3} \quad \forall i, j$$

$$\pi = \{\pi_i\} = \frac{1}{3} \quad \forall i$$



The three basic HMM problems

■ Problem 1: Probability Evaluation

- Given the observation sequence $O=\{o_1, o_2, o_3, o_4, \dots\}$ and a model $\lambda=\{A, B, \pi\}$, how do we efficiently compute $P(O|\lambda)$, the likelihood of the observation sequence given the model?
 - The solution to this question is given by the Forward and Backward procedures

■ Problem 2: Optimal State Sequence

- Given the observation sequence $O=\{o_1, o_2, o_3, o_4, \dots\}$ and a model λ , how do we choose a state sequence $Q=\{q_1, q_2, q_3, q_4, \dots\}$ that is optimal (i.e., best explains the data)?
 - The solution to this question is provided by the Viterbi algorithm

■ Problem 3: Parameter Estimation

- How do we adjust the parameters of the model $\lambda=\{A, B, \pi\}$ to maximize the likelihood $P(O|\lambda)$
 - The solution to this question is given by the Baum-Welch re-estimation procedure



Problem 1: Probability Evaluation (1)

- Our goal is to compute the likelihood of an observation sequence $O=\{o_1,o_2,o_3,o_4,\dots\}$ given a particular HMM model defined by $\lambda=\{A,B,\pi\}$
 - Computation of this probability involves enumerating every possible state sequence and evaluating the corresponding probability

$$P(O|\lambda) = \sum_{\forall Q} P(O|Q,\lambda)P(Q|\lambda)$$

- For a particular state sequence $Q=\{q_1,q_2,q_3,q_4,\dots\}$, the probability $P(O|Q,\lambda)$ is

$$P(O|Q,\lambda) = \prod_{t=1}^T P(o_t|q_t,\lambda) = \prod_{t=1}^T b_{q_t}(o_t)$$

- The probability of the state sequence Q is

$$P(Q|\lambda) = \pi_{q_1} a_{q_1q_2} a_{q_2q_3} \dots a_{q_{T-1}q_T}$$

- Merging these results, we obtain

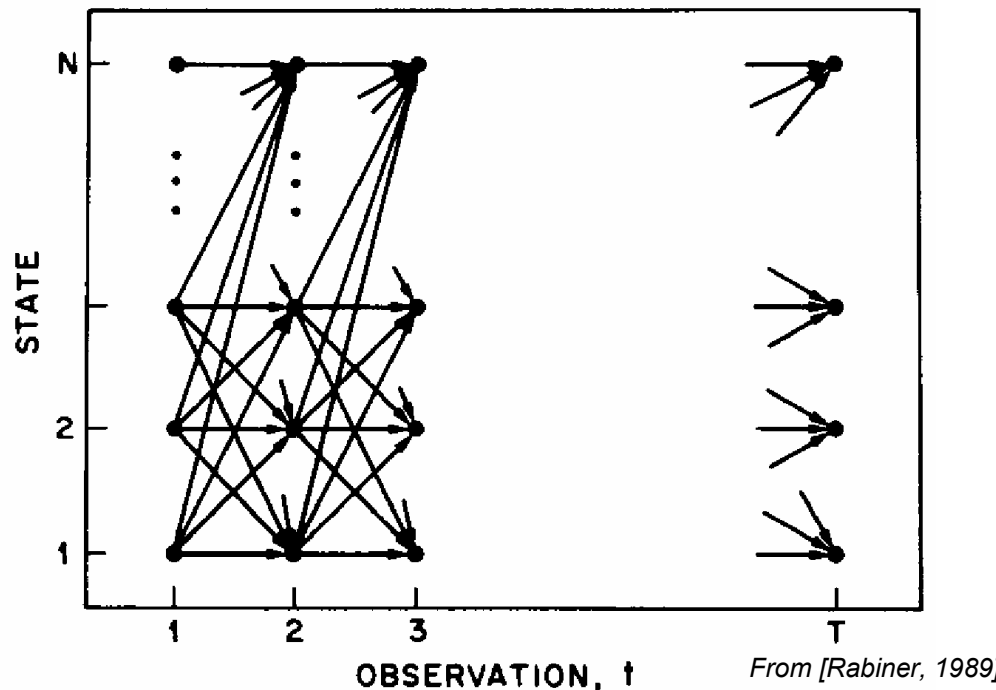
$$P(O|\lambda) = \sum_{q_1,q_2,\dots,q_T} \pi_{q_1} b_{q_1}(o_{q_1}) a_{q_1q_2} b_{q_2}(o_{q_2}) \dots a_{q_{T-1}q_T} b_{q_T}(o_{q_T})$$



Problem 1: Probability Evaluation (2)

■ Computational complexity

- With N^T possible state sequences, this approach becomes unfeasible even for small problems
 - For $N=5$ and $T=100$, the method would require the order of 10^{72} computations!
- Fortunately, the computation of $P(O|\lambda)$ has the lattice (or trellis) structure shown below, which lends itself to a very efficient implementation known as the **Forward procedure**



Problem 1: The Forward procedure

- Consider the following variable $\alpha_t(i)$ defined as

$$\alpha_t(i) = P(o_1, o_2, \dots, o_t, q_t = S_i | \lambda)$$

- which represents the probability of the observation sequence up to time t AND the state S_i at time t , given the model λ
- Computation of this variable can be efficiently performed by induction, as illustrated in the figure

- Initialization

$$\alpha_1(i) = \pi_i b_i(o_1)$$

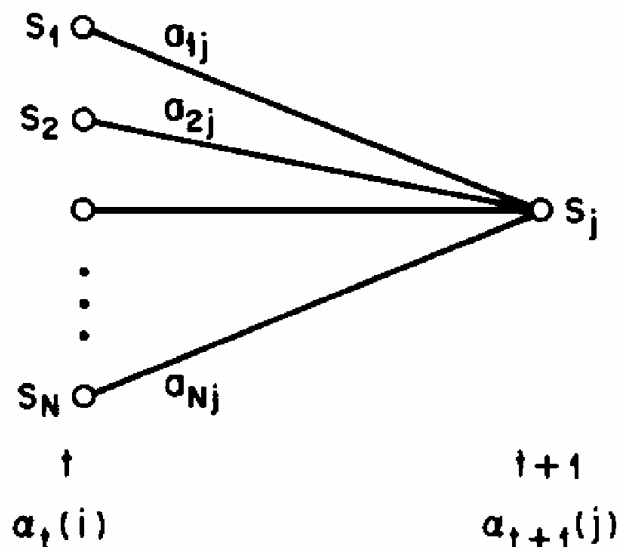
- Induction

$$\alpha_{t+1}(j) = \left[\sum_{i=1}^N \alpha_t(i) a_{ij} \right] b_j(o_{t+1}) \quad \begin{cases} 1 \leq t \leq T-1 \\ 1 \leq j \leq N \end{cases}$$

- Termination

$$P(O|\lambda) = \sum_{i=1}^N \alpha_T(i)$$

- As a result, computation of $P(O|\lambda)$ can be reduced from $2T \times N^T$ down to $N^2 \times T$ operations (from 10^{72} to 3000 for $N=5$, $T=100$)



From [Rabiner, 1989]



Problem 1: The Backward procedure

- In analogy to the forward procedure, consider the backward variable $\beta_t(i)$ defined as

$$\beta_t(i) = P(o_{t+1}, o_{t+2}, \dots, o_T | q_t = S_i, \lambda)$$

- $\beta_t(i)$ represents the probability of the partial observation sequence from $t+1$ to the end, given state S_i at time t , and the model λ
- As before, computation of $\beta_t(i)$ can be done through induction

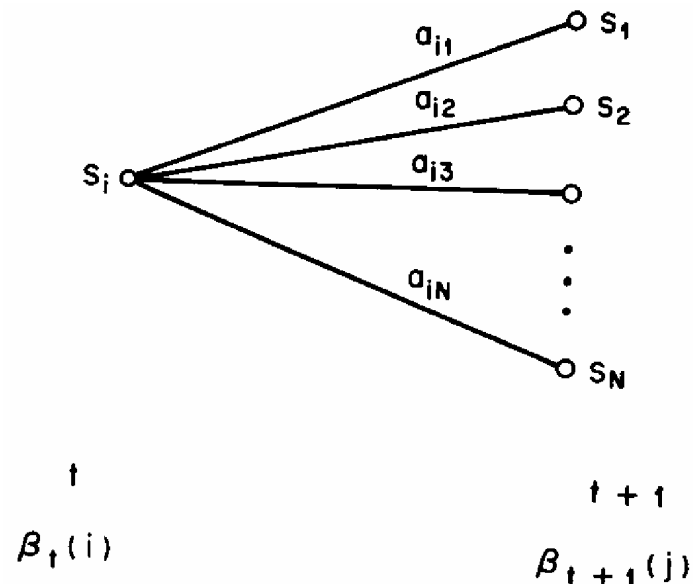
- Initialization

$$\beta_T(i) = 1 \text{ (arbitrarily)}$$

- Induction

$$\beta_t(i) = \sum_{j=1}^N a_{ij} b_j(o_{t+1}) \beta_{t+1}(j) \quad \begin{cases} t = T-1, T-2, 1 \\ 1 \leq i \leq N \end{cases}$$

- Similarly, this computation can be effectively performed in the order of $N^2 \times T$ operations



From [Rabiner, 1989]



Problem 2: Optimal State Sequence (1)

- **Finding the optimal state sequence is a more difficult problem than the estimation of $P(O|\lambda)$**
 - The main difficulty has to do with defining an optimality measure, since several criteria are possible
 - Finding the states q_t that are *individually* more likely at each time t
 - Finding the *single best* state sequence path (i.e., maximize the posterior $P(Q|O, \lambda)$)
 - As we will see in a minute, the second criterion is the one most widely used
 - However, we first optimize the first criterion as it allows us to define a variable that will be used later in the solution of Problem 3
- **As in the Forward-Backward procedures, we define a variable $\gamma_t(i)$**

$$\gamma_t(i) = P(q_t = S_i | O, \lambda)$$

- which represents the probability of being in state S_i at time t , given the observation sequence O and the model λ
- Using the definition of conditional probability, we can write

$$\gamma_t(i) = P(q_t = S_i | O, \lambda) = \frac{P(O, q_t = S_i | \lambda)}{P(O | \lambda)} = \frac{P(O, q_t = S_i | \lambda)}{\sum_{i=1}^N P(O, q_t = S_i | \lambda)}$$



Problem 2: Optimal State Sequence (2)

- Now, $\gamma_t(i)$ is equal to the product of $\alpha_t(i)$ and $\beta_t(i)$

$$\gamma_t(i) = \frac{P(O, q_t = S_i | \lambda)}{\sum_{i=1}^N P(O, q_t = S_i | \lambda)} = \frac{\alpha_t(i)\beta_t(i)}{\sum_{i=1}^N \alpha_t(i)\beta_t(i)}$$

- The individually most likely state q_t^* at each time is then

$$q_t^* = \underset{1 \leq i \leq N}{\operatorname{argmax}} [\gamma_t(i)] \quad \forall t = 1 \dots T$$

- The problem with choosing the individually most likely states is that the overall state sequence may not be valid
 - Consider a situation where the individually most likely states are $q_t = S_i$ and $q_{t+1} = S_j$, but the transition probability $a_{ij} = 0$
- To avoid this and other problems, it is common to look for the *single best* state sequence, at the expense of having sub-optimal individual states
 - This is accomplished with the **Viterbi algorithm**



Problem 2: The Viterbi algorithm (1)

- To find the single best state sequence we define yet another variable

$$\delta_t(i) = \max_{q_1 q_2 \dots q_{t-1} q_t} P[q_1 q_2 \dots q_{t-1} q_t = S_i, o_1 o_2 \dots o_t | \lambda]$$

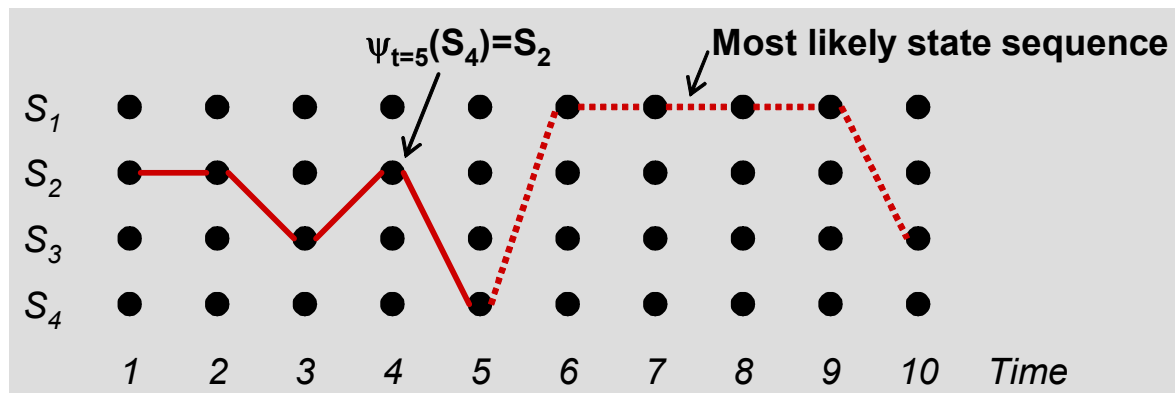
- which represents the highest probability along a single path that accounts for the first t observations and ends at state S_i
 - By induction, $\delta_{t+1}(j)$ can be computed as

$$\delta_{t+1}(j) = \max_i [\delta_t(i) a_{ij}] b_j(o_{t+1})$$

- To retrieve the state sequence, we also need to keep track of the state that maximizes $\delta_t(i)$ at each time t , which is done by constructing an array

$$\psi_{t+1}(j) = \operatorname{argmax}_{1 \leq i \leq N} [\delta_t(i) a_{ij}]$$

- $\psi_{t+1}(j)$ is the state at time t from which a transition to state S_j maximizes the probability $\delta_{t+1}(j)$



Problem 2: The Viterbi algorithm (2)

■ The Viterbi algorithm for finding the optimal state sequence becomes

■ Initialization

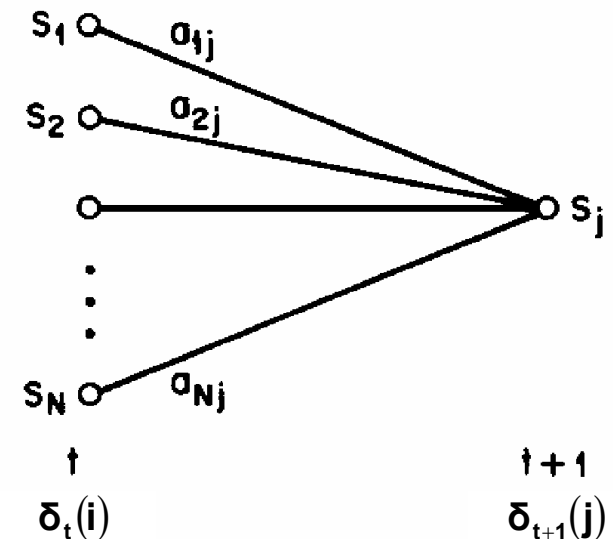
$$\begin{aligned}\delta_1(i) &= \pi_i b_i(o_1) \quad 1 \leq i \leq N \\ \psi_1(i) &= 0 \quad (\text{no previous states})\end{aligned}$$

■ Recursion

$$\left. \begin{aligned}\delta_t(j) &= \max_{1 \leq i \leq N} [\delta_{t-1}(i) a_{ij}] b_j(o_t) \\ \psi_t(j) &= \operatorname{argmax}_{1 \leq i \leq N} [\delta_{t-1}(i) a_{ij}]\end{aligned} \right\} \quad 2 \leq t \leq T; \quad 1 \leq j \leq N$$

■ Termination

$$\begin{aligned}P^* &= \max_{1 \leq i \leq N} [\delta_T(i)] \\ q_T^* &= \operatorname{argmax}_{1 \leq i \leq N} [\delta_T(i)]\end{aligned}$$



From [Rabiner, 1989]

- And the optimal state sequence can be retrieved by **backtracking**

$$q_t^* = \psi_{t+1}(q_{t+1}^*) \quad t = T-1, T-2, \dots, 1$$

- Notice that the Viterbi algorithm is similar to the Forward procedure, except that it uses a maximization over previous states instead of a summation

